COUNTING OCCURRENCES OF 3412 IN AN INVOLUTION

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Abstract

We study the generating function for the number of involutions on n letters containing exactly $r \geq 0$ occurrences of 3412. It is shown that finding this function for a given r amounts to a routine check of all involutions on $2r+1$ letters.

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1. Introduction

Permutations. Let $\pi \in S_n$ and $\tau \in S_m$ be two permutations. An *occurrence* of τ in π is a subsequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $(\pi(i_1), \ldots, \pi(i_m))$ is order-isomorphic to τ; in such a context, τ is usually called a *pattern*.

In the last decade much attention has been paid to the problem of finding the numbers $s_\sigma^r(n)$ for a fixed $r \geq 0$ and a given pattern τ (see [1, 2, 3, 5, 6, 7, 18, 20, 23, 25, 26, 27, 28, 29]). Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to patterns of length 3. Using two simple involutions (*reverse* and *complement*) on $\mathcal{S}_n$ it is immediate that with respect to being equidistributed, the six patterns of length three fall into the two classes $\{123, 321\}$ and $\{132, 213, 231, 312\}$. Noonan [22] proved that $s_{123}^1(n) = \frac{3}{n}\binom{2n}{n-3}$. A general approach to the problem was suggested by Noonan and Zeilberger [23]; they gave another proof of Noonan's result, and conjectured that $s_{123}^2(n) = \frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$ and $s_{132}^1(n) = \binom{2n-3}{n-3}$. The first conjecture was proved by Fulmek [10] and the latter conjecture was proved by Bóna in [6]. A conjecture of Noonan and Zeilberger states that $s_\sigma^r(n)$ is P-recursive in n for any r and τ. It was proved by Bóna [4] for $\sigma = 132$. Mansour and Vainshtein [20] suggested a new approach to this problem in the case $\sigma = 132$, which allows one to get an explicit expression for $s_{132}^r(n)$ for any given r. More precisely, they presented an algorithm that computes the generating function $\sum_{n\geq 0} s_{132}^r(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the symmetric group $\mathcal{S}_{2r}$. The algorithm has been implemented in C, and yields explicit results for $1 \leq r \leq 6$.

Involutions. An *involution* π is a permutation such that $\pi = \pi^{-1}$; let $\mathcal{I}_n$ denote the set of all n-involutions. An *even* (resp. *odd*) involution π is an involution such that the number of occurrences of the pattern 21 in π is an even (resp. odd) number.

Several authors have given enumerations of sets of involutions which avoid certain patterns. In [24] Regev provided an asymptotic formula for $\mathcal{I}_n(12\ldots k)$ and showed that $\mathcal{I}_n(1234)$ is enumerated by the nth Motzkin number $M_n = \sum_{i=0}^{[n/2]}\binom{n}{2i}C_i$. In [11] Gessel enumerated $\mathcal{I}_n(12\ldots k)$. In [13] Gouyou–Beauchamps gave an entirely bijective proof of some very nice exact formulas for $\mathcal{I}_n(12345)$ and $\mathcal{I}_n(123456)$.

Pattern-avoiding involutions have also been related with other combinatorial objects. Gire [12] has established a one-to-one correspondence between 1-2 trees having n edges and $\mathcal{S}_n(321, 3\bar{1}42)$ (this is the set of n-permutations avoiding patterns 321 and 231, except that the latter is allowed when it is a subsequence of the pattern 3142). On the other hand, Guibert [14] has established bijections between 1-2 trees having n edges and each of the sets $\mathcal{S}_n(231, 4\bar{1}32)$, $\mathcal{I}_n(3412)$, and $\mathcal{I}_n(4321)$ (and therefore with $\mathcal{I}_n(1234)$, by transposing the corresponding Young tableaux obtained by applying the Robinson-Schensted algorithm). Also, Guibert [14] has established a bijection between $\mathcal{I}_n(2143)$ and $\mathcal{I}_n(1243)$. More recently, Guibert, Pergola, and Pinzani [15] have given a one-to-one correspondence between 1-2 trees having n edges and $\mathcal{I}_n(2143)$. It follows that all these sets are enumerated by the nth Motzkin number M_n. It remains an open problem to prove the conjecture of Guibert (in [14]) that $\mathcal{I}_n(1432)$ is also enumerated by the nth Motzkin number M_n. This conjecture proved by Jaggard [16]. Recently, Egge [8] gave enumerations and generating functions for the number of involutions in $\mathcal{I}_n(3412)$ which avoid various sets of additional patterns. Egge and Mansour [9] refined Egge's results by studying generating functions for the number of even and odd involutions in $\mathcal{I}_n(3412)$, and they studied the generating functions for the number of even and odd involutions in $\mathcal{I}_n$ containing the pattern 3412 exactly once. For example, they proved the following theorem.

Theorem 1.1. (see [9, Proposition 7.7]) *The generating function for the number of involutions in $\mathcal{I}_n$ which contain the pattern* 3412 *exactly once is given by*

$$\frac{2x-1}{2x^2(1-x)} + \frac{1-2x-2x^2}{2x^2\sqrt{1-2x-3x^2}}.$$

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	2	4	9	21	51	127	323	835	2188	5798	15511
1	0	0	0	0	1	5	20	70	231	735	2289	7029	21384
2	0	0	0	0	0	0	1	7	37	165	671	2563	9375
3	0	0	0	0	0	0	1	4	17	63	236	877	3270
4	0	0	0	0	0	0	2	12	56	220	803	2783	9364
5	0	0	0	0	0	0	0	2	14	80	383	1658	6690
6	0	0	0	0	0	0	0	2	11	51	212	856	3402

Table 1. First values for the sequence $\mathcal{I}_{3412}^r(n)$ where $0 \leq n \leq 12$ and $0 \leq r \leq 6$

In this paper, we suggest a new approach to study the number of involutions which contain the pattern 3412 exactly r times, namely $\mathcal{I}_{3412}^r(n)$, which allows one to get an explicit expression for $\mathcal{I}_{3412}^r(n)$ for any given r (see Table 1). More precisely, we present an algorithm that computes the generating function $\mathcal{I}_r(x) = \sum_{n\geq 0} \mathcal{I}_{3412}^r(n)x^n$ for any $r \geq 0$. To get the result for a given r, the algorithm performs certain routine checks for each element of the involutions $\mathcal{I}_{2r+1}$. The algorithm has been implemented in C, and yields explicit results for $0 \leq r \leq 6$.

2. Recall definitions and preliminary results

2.1. **The case $r \geq 1$.** Fix $r \geq 1$. To any $\pi \in \mathcal{I}_n$ we assign a bipartite graph G_π in the following way. The vertices in one part of G_π, denoted V_1, are the entries of π, and the vertices of the second part, denoted V_4, are the occurrences of 3412 in π. Entry $i \in V_1$ is connected by an edge to occurrence $j \in V_4$ if i enters j. For example, let $\pi = 8\,2\,3\,13\,7\,6\,5\,1\,11\,12\,9\,10\,4\,14$, then π contains 2 occurrences of 3412, and the graph G_π is presented on Figure 1.

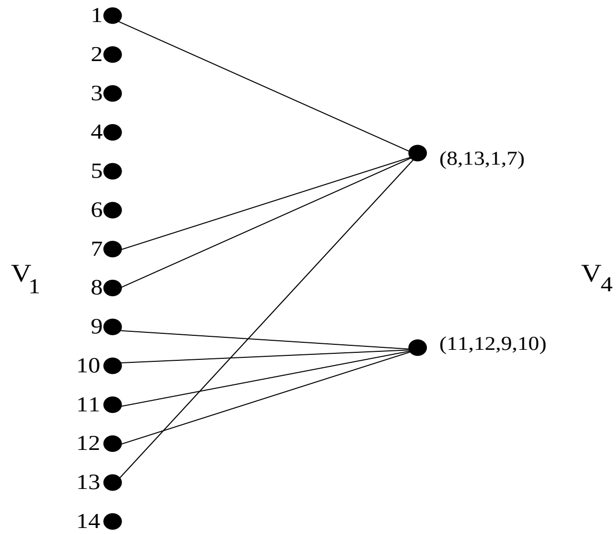

FIGURE 1. Graph G_π for $\pi = 8\,2\,3\,13\,7\,6\,5\,1\,11\,12\,9\,10\,4\,14$

Let $\widetilde{G}$ be an arbitrary connected component of G_π, and let $\widetilde{V}$ be its vertex set. We denote $\widetilde{V}_1 = \widetilde{V} \cap V_1$, $\widetilde{V}_4 = \widetilde{V} \cap V_4$, $t_1 = |\widetilde{V}_1|$, $t_4 = |\widetilde{V}_4|$.

Lemma 2.1. *For any connected component $\widetilde{G}$ of G_π one has $t_1 \leq 2t_4 + 2$.*

Proof. Assume to the contrary that the above statement is not true. Consider the smallest n for which there exists $\pi \in \mathcal{I}_n$ such that for some connected component $\widetilde{G}$ of G_π one has

$$t_1 > 2t_4 + 2. \tag{$*$}$$

Evidently, $\widetilde{G}$ contains more than one vertex, since otherwise $t_1 = 1$ and $t_4 = 0$, which contradicts $(*)$. Let l be the number of leaves in $\widetilde{G}$ (recall that a leaf is a vertex of degree 1). Clearly, all the leaves belong to $\widetilde{V}_1$; the degree of any other vertex in $\widetilde{V}_1$ is at least 2, while the degree of any vertex in $\widetilde{V}_4$ equals 4. Calculating the number of edges in $\widetilde{G}$ by two different ways, we get $l + 2(t_1 - l) \leq 4t_4$, which together with $(*)$ gives $l > 4$, so there exist at least five leaves in $\widetilde{V}_1$. In the case $r = 1$ there only one occurrence of length 4, so there no five leaves in V_1, therefore we can assume that $r \geq 2$.

Let $a \in \widetilde{V}_4$ and let us consider the following cases corresponding to the number of leaves in V_1 incident to a:

(i) If a incident to one leave $x \in V_1$ exactly, then the graph $\widetilde{G}'$ which obtained from $\widetilde{G}$ by deleting x and a satisfies $t_1' + 1 = t_1$ and $t_4' + 1 = t_4$, hence by $(*)$ we have that $t_1' > 2t_4' + 3$, a contradiction to the minimality of n.

(ii) If a incident to two leaves $x, y \in V_1$ exactly, then the graph $\widetilde{G}'$ which obtained from $\widetilde{G}$ by deleting x, y and a satisfies $t_1' + 2 = t_1$ and $t_4' + 1 = t_4$, hence by $(*)$ we have that $t_1' > 2t_4' + 2$, a contradiction to the minimality of n.

(iii) If there exist four leaves incident to a, then the graph $\widetilde{G}$ does not connect component, a contradiction.

(iv) By Cases(i)-(iii) we can assume that every vertex in V_4 incident to three leaves exactly. Consider $a, b \in V_4$ with the leaves $a^1, a^2, a^3 \in V_1$ and $b^1, b^2, b^3 \in V_1$ such that there exists a vertex $u \in V_1$ incident to a and b. Therefore, if we consider all the involutions π of length at most 7 with two occurrences a and b of 3412 and the corresponding graph G_π is connected

component, then we see that each involution has two occurrences $x_1x_2x_3x_4$ and $y_1y_2y_3y_4$ of 3412 such that $|\{x_1,x_2,x_3,x_4\}\cap\{y_1,y_2,y_3,y_4\}| = 2$, a contradiction for that every vertex in V_4 incident to three leaves exactly (more precisely, in $\mathcal{I}_k$, $k = 0,1,2,\dots,5$, there no involutions which contain 3412 exactly twice; in $\mathcal{I}_6$ there exists one involution which contain 3412 exactly twice, namely 351624; and in $\mathcal{I}_7$ we have 7 involutions which contain 3412 exactly twice, namely 1462735, 3516247, 3517264, 3614725, 3617524, 4261735, and 4631725. Each of these involutions has two occurrences $x_1x_2x_3x_4$ and $y_1y_2y_3y_4$ of 3412 such that there $|\{x_1,x_2,x_3,x_4\}\cap\{y_1,y_2,y_3,y_4\}| = 2$).

Hence, using Cases (i)-(iv) we get the desired result. □

Denote by G_π^1 the connected component of G_π containing entry 1. Let $\pi(i_1),\dots,\pi(i_s)$ be the entries of π belonging to G_π^1, and let $\sigma = \sigma_\pi \in \mathcal{I}_s$ be the corresponding involution (σ_π is an involution for any involution π) . We say that $\pi(i_1),\dots,\pi(i_s)$ is the *kernel* of π and denote it $\ker\pi$; σ is called the *shape* of the kernel, or the *kernel shape*, s is called the *size* of the kernel, and the number of occurrences of 3412 in $\ker\pi$ is called the *capacity* of the kernel. For example, for $\pi = 8\,2\,3\,13\,7\,6\,5\,1\,11\,12\,9\,10\,4\,14$ as above, the kernel equals $8\,13\,1\,4$, its shape is 3412, the size equals 4, and the capacity equals 1.

The following statement is implied immediately by Lemma 2.1.

Theorem 2.2. *Fix $r \geq 1$. Let $\pi \in \mathcal{I}_n$ contain exactly r occurrences of 3412, then the size of the kernel of π is at most $2r+2$.*

We say that ρ is a *kernel involution* if it is the kernel shape for some involution π. Evidently ρ is a kernel involution if and only if $\sigma_\rho = \rho$ is an involution.

Let $\rho \in \mathcal{I}_s$ be an arbitrary kernel involution. We denote by $\mathcal{I}(\rho)$ the set of all the involutions of all possible sizes whose kernel shape equals ρ. For any $\pi \in \mathcal{I}(\rho)$ we define the *kernel cell decomposition* as follows. The number of cells in the decomposition equals $s \times s$. Let $\ker\pi = \pi(i_1),\dots,\pi(i_s)$; the *cell* $C_{m\ell} = C_{m\ell}(\pi)$ for $1 \leq \ell \leq s$ and $1 \leq m \leq s$ is defined by

$$C_{m\ell}(\pi) = \{\pi(j)|\, i_\ell < j < i_{\ell+1},\ \pi(i_{\rho^{-1}(m)}) < \pi(j) < \pi(i_{\rho^{-1}(m+1)})\},$$

where $i_{s+1} = n+1$ and $\alpha_{n+1} = n+1$ for any $\alpha \in \mathcal{I}_n$. If π coincides with ρ itself, then all the cells in the decomposition are empty. An arbitrary permutation in $\mathcal{I}(\rho)$ is obtained by filling in some of the cells in the cell decomposition. A cell C is called *infeasible* if the existence of an entry $a \in C$ would imply an occurrence of 3412 that contains a, two other entries $x, y \in \ker\pi$, and another entry z of π. Clearly, all infeasible cells are empty for any $\pi \in \mathcal{I}(\rho)$. All the remaining cells are called *feasible*; a feasible cell may, or may not, be empty. The set of all feasible cells are distributed into three subsets:

(i) *Free-cells*; a free-cell is a feasible cell may, or may not, be contain an occurrence of the pattern 12, or contain an occurrence of the pattern 21;
(ii) *Diagonal-decreasing-cells*; a diagonal-decreasing-cell is a feasible cell $C_{ii} = \{\pi_{j_1},\dots,\pi_{j_d}\}$ such that $\pi_{j_1} > \cdots > \pi_{j_d}$ where $j_1 < \cdots < j_d$;
(iii) *Decreasing-cells*; a decreasing-cell is a feasible cell $C_{ij} = \{\pi_{j_1},\dots,\pi_{j_d}\}$ with $i \neq j$ such that $\pi_{j_1} > \cdots > \pi_{j_d}$ where $j_1 < \cdots < j_d$.

Consider the involution $\pi = 8\,2\,3\,13\,7\,6\,5\,1\,11\,12\,9\,10\,4\,14$. The kernel of π equals $8\,13\,1\,4$, its shape is 3412. The cell decomposition of π contains four feasible cells: $C_{11} = \{2,3\}$, $C_{22} = \{7,6,5\}$, $C_{33} = \{11,12,9,10\}$, and $C_{44} = \{14\}$, see Figure 2. All the other cells are infeasible; for example, C_{21} is infeasible, since if $a \in C_{21}$, then $a\pi'_{i_2}\pi'_{i_3}\pi'_{i_4}$ is an occurrence of 3412 for any π' whose kernel is of shape 3412. The cells C_{11}, C_{33}, and C_{44} are free-cells, and C_{22} is diagonal-decreasing-cell.

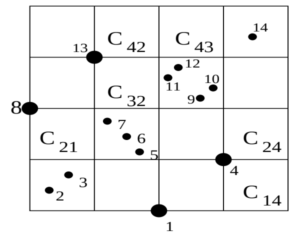

FIGURE 2. Kernel cell decomposition for $\pi \in \mathcal{I}(3412)$.

Given a cell C_{ij} in the kernel cell decomposition, all the kernel entries can be positioned with respect to C_{ij}. We say that $x = \pi(i_k) \in \ker \pi$ lies *below* C_{ij} if $\rho(k) < i$, and *above* C_{ij} if $\rho(k) \geq i$. Similarly, x lies to the *left* of C_{ij} if $k < j$, and to the *right* of C_{ij} if $k \geq j$. As usual, we say that x lies to the *southwest* of C_{ij} if it lies below C_{ij} and to the left of it; the other three directions, northwest, southeast, and northeast, are defined similarly.

The following statement plays a crucial role in our considerations.

Lemma 2.3. *Let $\pi \in \mathcal{I}(\rho)$.*

(i) *C_{ij} is infeasible cell if and only if C_{ji} is infeasible cell.*

(ii) *C_{ij} is feasible cell with $i \neq j$ if and only if C_{ij} and C_{ji} are decreasing-cells.*

(iii) *Let C_{ii} be a feasible cell; if there exits northwest or southeast of C_{ii} an occurrence of the pattern 12 in the kernel shape ρ, then C_{ii} is diagonal-decreasing-cell, otherwise C_{ii} is free-cell.*

Proof. (i) Holds immediately by the fact that π is an involution.

(ii) Let $i \neq j$. Using the fact that π is an involution we get that C_{ij} contains an occurrence of the pattern 12 if and only if C_{ji} contains an occurrence of the pattern 12. Thus, if C_{ij} contains an occurrence of the pattern 12, namely xy, then C_{ji} contains an occurrence uv of the pattern 12, so $xyuv$ is an occurrence of the pattern 3412, a contradiction. Hence, C_{ij} is feasible cell if and only if C_{ij} and C_{ji} are decreasing-cells.

(iii) Let C_{ii} be a feasible cell such that there exists northwest of C_{ii} an occurrence of the pattern 12 in the kernel shape ρ, namely xy. So, if C_{ii} contains an occurrence of the pattern 12 in C_{ii}, say uv, then $xyuv$ is an occurrence of the pattern 3412. Hence, C_{ii} is diagonal-decreasing-cell. $\square$

Similarly as the arguments in the proof of Lemma 2.3 we have the following result.

Lemma 2.4. *Let $a < b$.*

(i) *If C_{ia} and C_{ib} are decreasing-cells then every entry of C_{ia} is greater than every entry of C_{ib};*

(ii) *If C_{ai} and C_{bi} are decreasing-cells then the entries of C_{bi} are lie northwest of the entries of C_{ib};*

(iii) *If C_{ab} is decreasing-cell, then all the cells C_{ij}, with $i \neq j$, which are lie northeast of C_{ab} and C_{ba} are infeasible cells.*

As a consequence of Lemmas 2.3 and 2.4, we get the following result. Let us define a partial order $\prec$ on the set of all free-cells by saying that $C_{ml} \prec C_{m'\ell'} \neq C_{m\ell}$ if $m \leq m'$ and $\ell \leq \ell'$. Similarly, we define a partial order $\prec'$ and $\prec''$ on the set of all diagonal-decreasing-cells and decreasing-cells, respectively.

Lemma 2.5. *$\prec$, $\prec'$, and $\prec''$ are linear orders.*

Lemmas 2.3–2.5 yield immediately the following two results.

Theorem 2.6. *Let $\widetilde{G}$ be a connected component of G_π distinct from G_π^1. Then all the vertices in $\widetilde{V}_1$ belong to the same feasible cell in the kernel cell decomposition of π.*

Let $F(\rho)$ (respectively; $DD(\rho)$ and $D(\rho)$) be the set of all free-cells (respectively; diagonal-decreasing-cells and decreasing-cells which lie above the diagonal) in the kernel cell decomposition corresponding to involutions in $\mathcal{I}(\rho)$ and let $f(\rho) = |F(\rho)|$ (respectively; $dd(\rho) = |DD(\rho)|$ and $d(\rho) = |D(\rho)|$). We remark that, by Lemma 2.3 and Lemma 2.4 we get that $d(\rho)$ is a nonnegative integer number. We denote the cells in $F(\rho)$ by $C^1, \dots, C^{f(\rho)}$ (respectively; $DD^1, \dots, DD^{dd(rho)}$ and $D^1, \dots, D^{d(\rho)}$) in such a way that $C^i \prec C^j$ (respectively; $DD^i \prec' DD^j$ and $D^i \prec'' D^j$) whenever $i < j$.

Theorem 2.7. *For any given sequence $\alpha_1, \dots, \alpha_{f(\rho)}$ of arbitrary involutions, and two sequences $\beta_1, \dots, \beta_{dd(\rho)}$ and $\gamma_1, \dots, \gamma_{d(\rho)}$ of arbitrary decreasing involutions (a decreasing involution of length n is the involution $n(n-1)\dots 1$), there exists $\pi \in \mathcal{I}(\rho)$ such that the content of the free-cell C^i is order-isomorphic to α_i, the content of the diagonal-decreasing-cell DD^j is β_j, and the content of the decreasing-cell D^j is γ_j.*

2.2. **The case** $r = 0$**.** First of all, let us describe the cell block decomposition of an involution $\pi \in \mathcal{I}_n(3412)$ as follows.

Proposition 2.8. (see [8, Proposition 2.8]) *Let $\pi \in I_n(3412)$. Then one of the following holds:*
(i) $\pi = (1, \alpha)$ *where* $(\alpha_1 - 1, \dots, \alpha_{n-1} - 1) \in I_{n-1}(3412)$,
(ii) *there exists t, $2 \le t \le n$, such that $\pi = (t, \alpha, 1, \beta)$ where $(\alpha_1 - 1, \dots, \alpha_{t-2} - 1) \in I_{t-2}(3412)$ and $(\beta_1 - t, \dots, \beta_{n-t} - t) \in I_{n-t}(3412)$.*

Thus, the kernel cell decomposition of $\pi \in \mathcal{I}_n(3412)$ can be defined as follows. There are two kernel shapes $\rho^1 = 1$ and $\rho^2 = 21$. In the case ρ^1 there exists only one cell which is free-cell, and in the case ρ^2 there exist four cells: two are infeasible cells (C_{21} and C_{12}), and the others are free-cells (C_{11} and C_{22}), see Figure 3.

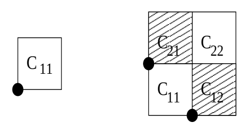

FIGURE 3. Kernel cell decomposition for $\pi \in \mathcal{I}(1) \cup \mathcal{I}(21)$.

3. Main Theorem and explicit results

Let ρ be a kernel involution, and let $s(\rho)$, $c(\rho)$, $f(\rho)$, $dd(\rho)$, and $d(\rho)$ be the size of ρ, the capacity of ρ, and the number of free-cells, diagonal-decreasing-cells, decreasing-cells in the cell decomposition associated with ρ, respectively. Denote by K the set of all kernel involutions, and by K_t the set of all kernel shapes for involutions in $\mathcal{I}_t$. The main result of this note can be formulated as follows.

Theorem 3.1. *For any $r \ge 1$,*

$$\mathcal{I}_r(x) = \sum_{\rho \in K_{2r+2}} \left(\frac{x^{s(\rho)}}{(1-x^2)^{d(\rho)}(1-x)^{dd(\rho)}} \sum_{r_1 + \cdots + r_{f(\rho)} = r - c(\rho)} \prod_{j=1}^{f(\rho)} \mathcal{I}_{r_j}(x) \right), \tag{3.1}$$

where $r_j \ge 0$ for $1 \le j \le f(\rho)$.

Proof. For any $\rho \in K$, denote by $\mathcal{I}_r^{\rho}(x)$ the generating function for the number of involutions in $\pi \in \mathcal{I}_n \cap \mathcal{I}(\rho)$ containing exactly r occurrences of 3412. Evidently, $\mathcal{I}_r(x) = \sum_{\rho \in K} \mathcal{I}_r^{\rho}(x)$. To find $\mathcal{I}_r^{\rho}(x)$, recall that the kernel of any π as above contains exactly $c(\rho)$ occurrences of 3412. The remaining $r - c(\rho)$ occurrences of 3412 are distributed between the free-cells of the kernel cell decomposition of π. By Theorem 2.6, each occurrence of 3412 belongs entirely to one free-cell. Besides, it follows from

Theorem 2.7, that occurrences of 3412 in different free-cells do not influence one another. Also, by Lemmas 2.3 and 2.4 the diagonal-decreasing-cells do not influence one another, and for decreasing cell C_{ij} we have that $|C_{ij}| = |C_{ji}|$ where C_{ij} contains arbitrary decreasing involution. Therefore,

$$\mathcal{I}_r^\rho(x) = \frac{x^{s(\rho)}}{(1-x^2)^{d(\rho)}(1-x)^{dd(\rho)}} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho)} \prod_{j=1}^{f(\rho)} \mathcal{I}_{r_j}(x),$$

and we get the expression similar to (3.1) with the only difference that the outer sum is over all $\rho \in K$. However, if $\rho \in K_t$ for $t > 2r+2$, then by Theorem 2.2, $c(\rho) > r$, and hence $\mathcal{I}_r^\rho(x) \equiv 0$. □

Theorem 3.1 provides a finite algorithm for finding $\mathcal{I}_r(x)$ for any given $r \geq 0$, since we have to consider all involutions in $\mathcal{I}_{2r+2}$, and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition.

Proposition 3.2. *Let $\psi^0 = 21$, $\psi^1 = 3412$, $\psi^2 = 351624$, and*

$$\psi^r = 3\ 5\ 1\ 7\ 2\ 9\ 4\ 11 \ldots (2r+1)\ (2r-4)\ (2r+2)\ (2r-2)\ (2r)$$

for all $r \geq 3$. Then the only kernel involution of capacity $r \geq 0$ and size $2r+2$ is ψ^r. Its contribution to $\mathcal{I}_r(x)$ equals $\frac{x^{2r+2}}{(1-x)^r}\mathcal{I}_0^{r+2}(x)$.

This proposition is proved easily by induction, similarly to Lemma 1. The feasible cells in the corresponding cell decomposition is: C_{ii} is free-cell if $i = 1, 3, 5, \ldots, 2r+1, 2r+2$, C_{ii} is diagonal-decreasing-cell if $i = 2, 4, \ldots, 2r$, and all the other cells are infeasible cells, hence the contribution to $\mathcal{I}_r(x)$ is as described. By the above proposition, it suffices to search only involutions in $\mathcal{I}_{2r+1}$. Below we present several explicit calculations.

3.1. **The case $r = 0$.** Let us start from the case $r = 0$. Observe that (3.1) remains valid for $r = 0$, provided the left hand side is replaced by $\mathcal{I}_r(x) - 1$; subtracting 1 here accounts for the empty involution. Also, by Subsection 2.2 we have only two shapes $\rho^1 = 1$ and $\rho^2 = 21$ with $s(\rho^1) = 1$, $c(\rho^1) = 0$, $f(\rho^1) = 1$, $dd(\rho^1) = d(\rho^1) = 0$, $s(\rho^2) = 2$, $c(\rho^2) = 0$, $f(\rho^2) = 2$, and $dd(\rho^2) = d(\rho^2) = 0$. Therefore, we get

$$\mathcal{I}_0(x) - 1 = x\mathcal{I}_0(x) + x^2\mathcal{I}_0^2(x).$$

Corollary 3.3. (see [14, Rem. 4.28]) *The generating function for the number of involutions which avoid* 3412 *is given by*

$$\mathcal{I}_0(x) = \frac{1 - x - \sqrt{1-2x-3x^2}}{2x^2}.$$

3.2. **The case $r = 1$.** Let now $r = 1$. The involutions in $\mathcal{I}_4$ are exhibited only one kernel shape distinct from ρ^1 and ρ^2 which is $\rho^3 = 3412$, whose contribution equals $\frac{x^4}{1-x}\mathcal{I}_0^3(x)$ (see Figure 4).

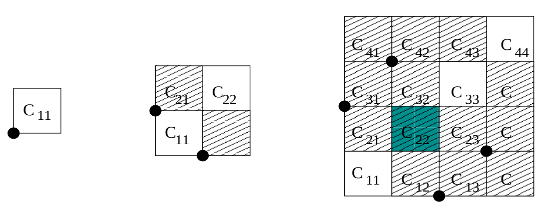

Figure 4. Kernel cell decomposition for $\pi \in \mathcal{I}(1) \cup \mathcal{I}(21) \cup \mathcal{I}(3412)$.

Therefore, (3.1) amounts to $\mathcal{I}_1(x) = x\mathcal{I}_1(x) + 2x^2\mathcal{I}_0(x)\mathcal{I}_1(x) + \frac{x^4}{1-x}\mathcal{I}_0^3(x)$, and we get the following result.

Corollary 3.4. *The generating function for the number of involutions which contain* 3412 *exactly once is given by*

$$\mathcal{I}_1(x) = -\frac{1-2x}{2x^2(1-x)} + \frac{1-2x-2x^2}{2x^2}\sqrt{1-2x-3x^2}^{\,-1}.$$

3.3. **The case $r=2$.** Let $r=2$. We have to check the kernel shapes of involutions in $\mathcal{I}_6$. Exhaustive search adds one new shape to the previous list; this is $\rho^4 = 351624$. Calculation of the parameters s, c, f, d, dd is straightforward, and we get

Corollary 3.5. *The generating function for the number of involutions which contain* 3412 *exactly twice is given by*

$$\mathcal{I}_2(x) = \frac{1-2x}{2x^2(1-x)} - \frac{1-6x+8x^2+8x^3-15x^4-2x^5+4x^6}{2x^2(1-x)^2}\sqrt{1-2x-3x^2}^{\,-3}.$$

3.4. **The cases $r=3,4,5,6,7$.** Let $r=3,4,5,6$; exhaustive search in $\mathcal{I}_8$, $\mathcal{I}_{10}$, $\mathcal{I}_{12}$, and $\mathcal{I}_{14}$ reveals 2, 5, 12, 25, 48, and 100 new kernel shapes, respectively, and we get

Corollary 3.6. *Let $r=3,4,5,6,7$. Then the generating function for the number of involutions which contain* 3412 *exactly r times is given by*

$$\mathcal{I}_r(x) = \frac{1}{2x^2}F_r(x) + \frac{1}{2x^2}G_r(x)\sqrt{1-2x-3x^2}^{\,1-2r},$$

where

$$\begin{aligned}
(1-x^2)F_3(x) &= -(1-2x)(1+x+x^2),\\
(1-x^2)F_4(x) &= -1+3x+4x^2-8x^3-2x^4,\\
(1-x^2)F_5(x) &= 3-7x-7x^2+12x^3+6x^4,\\
(1-x^2)^2F_6(x) &= -5+9x+21x^2-25x^3-34x^4+16x^5+24x^6-2x^7-2x^8,\\
(1-x^2)^2F_7(x) &= 7-11x-28x^2+20x^3+54x^4-2x^5-46x^6+2x^8,
\end{aligned}$$

and

$$\begin{aligned}
(1-x)^2G_3(x) &= 1-8x+18x^2+x^2-29x^4-12x^5+14x^6+41x^7+2x^8-18x^9,\\
(1-x)^4G_4(x) &= 1-14x+71x^2-124x^3-166x^4+874x^5-624x^6-1332x^7+1909x^8\\
&\quad +426x^9-1585x^{10}+292x^{11}+400x^{12}-126x^{13},\\
(1-x)^4G_5(x) &= -3+46x-267x^2+627x^3+134x^4-3321x^5+3954x^6+5214x^7\\
&\quad -11775x^8-2186x^9+14525x^{10}-1701x^{11}-8824x^{12}+1537x^{13}\\
&\quad +2594x^{14}-216x^{15}-324x^{16},\\
(1-x)^6G_6(x) &= 5-94x+712x^2-2582x^3+3124x^4+8364x^5-31620x^6+15464x^7\\
&\quad +77508x^8-107098x^9-76814x^{10}+214160x^{11}+5782x^{12}-231050x^{13}\\
&\quad +62700x^{14}+146176x^{15}-65653x^{16}-50328x^{17}+29646x^{18}\\
&\quad +6462x^{19}-5346x^{20}+486x^{21},\\
(1-x)^6G_7(x) &= -7+144x-1210x^2+5020x^3-8206x^4-12180x^5+69464x^6\\
&\quad -54210x^7-181468x^8+315366x^9+239852x^{10}-779338x^{11}\\
&\quad -124766x^{12}+1226006x^{13}-168810x^{14}-1272344x^{15}+418555x^{16}\\
&\quad +813368x^{17}-373802x^{18}-279554x^{19}+153648x^{20}+37188x^{21}\\
&\quad -23166x^{22}+486x^{23}.
\end{aligned}$$

4. Further results

As an easy consequence of Theorem 3.1 and Corollary 3.3 we get the following result.

Corollary 4.1. *Let $r \geq 0$, then $\mathcal{I}_r(x)$ is a rational function in the variables x and $\sqrt{1-2x-3x^2}$.*

Another direction would be to match the approach of this note with the previous results on even (odd) permutations which contain 132 exactly r times (see [19]). We denote the generating function for the number of even (resp. odd) involutions in $\mathcal{I}_n$ which contain r occurrences of 3412 by $E_r(x)$ (resp. $O_r(x)$) (see Table 2).

$r\backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	2	3	11	31	71	155	379	1051	2971	8053
1	0	0	0	0	1	5	14	30	82	320	1213	3895	11141
2	0	0	0	0	0	0	0	0	11	95	439	1463	4407
3	0	0	0	0	0	0	1	4	11	29	104	396	1486
4	0	0	0	0	0	0	0	0	14	108	321	1612	4782
5	0	0	0	0	0	0	0	0	6	60	275	878	2247
6	0	0	0	0	0	0	0	0	1	21	122	446	1504

Table 2. Number of even involutions in $\mathcal{I}_n$ which contain 3412 exactly r times where $0 \leq n \leq 12$ and $0 \leq r \leq 6$.

We define $N_r(x) = E_r(x) - O_r(x)$ for any r, that is,

$$N_r(x) = \sum_{n\geq 0} \sum_{\pi\in\mathcal{I}_n \text{ contains 3412 exactly } r \text{ times}} (-1)^{21(\pi)} x^n,$$

where $21(\pi)$ is the number of occurrences of the pattern 21 in π. Using the arguments in the proof of Theorem 3.1 we get the following result.

Theorem 4.2. *For any $r \geq 0$,*

$$N_r(x) - \delta_{r,0} = \sum_{\rho\in K_{2r+2}} \left[\frac{(-1)^{21(\rho)} x^{s(\rho)} (1+x)^{dd(\rho)}}{(1+x^2)^{d(\rho)+dd(\rho)}} \sum_{r_1+\cdots+r_{f(\rho)}=r-c(\rho)} \prod_{j=1}^{f(\rho)} N_{r_j}(x) \right], \tag{4.1}$$

where $r_j \geq 0$ for $1 \leq j \leq f(\rho)$, $\delta_{0,0} = 1$, and $\delta_{r,0} = 0$ for all $r \geq 1$.

As an easy consequence of Theorem 4.2 we get the following result.

Corollary 4.3. *Let $r \geq 0$, then $N_r(x)$ is a rational function in the variables x and $\sqrt{1-2x+5x^2}$.*

Clearly,

$$E_r(x) = \frac{1}{2}(\mathcal{I}_r(x) + N_r(x)) \text{ and } O_r(x) = \frac{1}{2}(\mathcal{I}_r(x) - N_r(x)), \tag{4.2}$$

for all $r \geq 0$. Hence, Corollary 4.1 and Corollary 4.3 give the following result.

Corollary 4.4. *For any $r \geq 0$, the generating functions $E_r(x)$ and $O_r(x)$ are rational functions in the variables x, $\sqrt{1-2x-3x^2}$, and $\sqrt{1-2x+5x^2}$.*

Hence, Theorem 4.2 and Theorem 3.1 provide a finite algorithm for finding $\mathcal{I}_r(x)$, $N_r(x)$, $E_r(x)$, and $O_r(x)$ for any given $r \geq 0$, since we have to consider all involutions in $\mathcal{I}_{2r+2}$, and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition which holds as easily consequence of Proposition 3.2.

Proposition 4.5. *Let $\psi^0 = 21$, $\psi^1 = 3412$, $\psi^2 = 351624$, and*

$$\psi^r = 3\ 5\ 1\ 7\ 2\ 9\ 4\ 11 \ldots (2r+1)\ (2r-4)\ (2r+2)\ (2r-2)\ (2r)$$

for all $r \geq 3$. Then the only kernel involution of capacity $r \geq 0$ and size $2r+2$ is ψ^r. Its contribution to $N_r(x)$ equals $(-1)^{r+1}\frac{x^{2r+2}(1+x)^r}{(1+x^2)^r}N_0^{r+2}(x)$.

By the above proposition, it suffices to search only involutions in $\mathcal{I}_{2r+1}$. Similarly as our calculations for $\mathcal{I}_r(x)$ where $r = 0,1,2,3,4,5,6,7$ with using Theorem 4.2 we get the following result.

Corollary 4.6. *Let $r = 0,1,2,3,4,5,6,7$. Then the generating function $N_r(x)$ is given by*

$$N_r(x) = \frac{1}{2x^2}P_r(x) + \frac{1}{2x^2}Q_r(x)\sqrt{1-2x+5x^2}^{\,1-2r},$$

where

$$\begin{aligned}
P_0(x) &= x-1,\\
(x^2+1)P_1(x) &= (x+1)(2x^2-2x+1),\\
(x^2+1)^2P_2(x) &= (x-1)(2x^2-2x+1)(1+x)^2,\\
(x^2+1)^3P_3(x) &= (2x^2-2x+1)(x^6+x^5+3x^4-2x^3-x^2+x+1),\\
(x^2+1)^4P_4(x) &= (1-x^2)(2x^8-2x^7+10x^6+x^5+15x^4-12x^3+12x^2-3x+1),\\
(x^2+1)^5P_5(x) &= -4x^{13}-14x^{11}+5x^{10}-33x^9+29x^8-16x^7+34x^6-42x^5+14x^4+6x^3\\
&\quad -15x^2+7x-3,\\
(x^2+1)^6P_6(x) &= -6x^{16}+6x^{15}-44x^{14}+58x^{13}-128x^{12}+163x^{11}-195x^{10}+271x^9-221x^8\\
&\quad +188x^7-170x^6+160x^5-82x^4+27x^3+9x^2-9x+5,\\
(x^2+1)^7P_7(x) &= -14x^{18}+12x^{17}-80x^{16}+94x^{15}-176x^{14}+212x^{13}-188x^{12}+247x^{11}\\
&\quad -157x^{10}+35x^9-51x^8+28x^7+62x^6-142x^5+102x^4-49x^3-3x^2+11x-7,
\end{aligned}$$

and

$$\begin{aligned}
Q_0(x) &= 1,\\
(x^2+1)Q_1(x) &= (x^2-1)(4x^2-2x+1),\\
(x^2+1)^2Q_2(x) &= (22x^6-58x^5+69x^4-48x^3+22x^2-6x+1)(1+x)^2,\\
(x^2+1)^3Q_3(x) &= (x-1)\big(100x^{12}-18x^{11}+323x^{10}-507x^9+491x^8-182x^7+52x^6-14x^5\\
&\quad +46x^4-34x^3+19x^2-5x+1\big),\\
(x^2+1)^4Q_4(x) &= -(1+x)\big(650x^{16}-1880x^{15}+5992x^{14}-9143x^{13}+13671x^{12}-19666x^{11}\\
&\quad +26606x^{10}-28683x^9+24771x^8-16778x^7+9158x^6-3969x^5+1385x^4-374x^3\\
&\quad +78x^2-11x+1\big),\\
(x^2+1)^5Q_5(x) &= (1-x)\big(5000x^{21}-4650x^{20}+24624x^{19}-25585x^{18}+76987x^{17}-95269x^{16}\\
&\quad +127936x^{15}-140244x^{14}+169896x^{13}-159580x^{12}+119898x^{11}-51878x^{10}\\
&\quad -84x^9+26302x^8-26778x^7+17822x^6-8604x^5+3270x^4-940x^3+209x^2\\
&\quad -31x+3\big),
\end{aligned}$$

$$\begin{aligned}
(x^2+1)^6Q_6(x) &= -43750x^{27}+133750x^{26}-542500x^{25}+1329674x^{24}-2984612x^{23}\\
&+5378699x^{22}-8590394x^{21}+12236909x^{20}-15828644x^{19}+18229621x^{18}\\
&-19177696x^{17}+18659837x^{16}-17024788x^{15}+14266232x^{14}-10700428x^{13}\\
&+6908636x^{12}-3700402x^{11}+1527142x^{10}-395516x^{9}-27686x^{8}+101584x^{7}\\
&-68679x^{6}+30486x^{5}-10181x^{4}+2580x^{3}-493x^{2}+64x-5,\\
(x^2+1)^7Q_7(x) &= -481250x^{31}+1658750x^{30}-5844500x^{29}+14332172x^{28}-29824134x^{27}\\
&+52203592x^{26}-78380980x^{25}+104774831x^{24}-124983968x^{23}+132048678x^{22}\\
&-122776812x^{21}+101431782x^{20}-72478438x^{19}+39230434x^{18}-4374004x^{17}\\
&-25236483x^{16}+42491126x^{15}-44337242x^{14}+34831062x^{13}-21298364x^{12}\\
&+9941638x^{11}-3111220x^{10}+199166x^{9}+519349x^{8}-425180x^{7}+212566x^{6}\\
&-78950x^{5}+22882x^{4}-5138x^{3}+874x^{2}-102x+7.
\end{aligned}$$

For example, by Corollary 3.3 and Corollary 4.6 for $r=0$ we have that the generating function for the number even involutions which avoid 3412 is given by

$$E_0(x)=\frac{\sqrt{1-2x+5x^2}-\sqrt{1-2x-3x^2}}{4x^2},$$

and the generating function for the number of odd involution which avoid 3412 is given by

$$O_0(x)=\frac{2-2x-\sqrt{1-2x+5x^2}-\sqrt{1-2x-3x^2}}{4x^2}.$$

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